Coherent state of the effective mass harmonic oscillator

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Abstract

We construct coherent state of the effective mass harmonic oscillator and examine some of it's properties. In particular closed form expressions of coherent states for different choices of the mass function are obtained and it is shown that such states are not in general $x - p$ uncertainty states. We also compute the associated Wigner functions.

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1 Introduction

Schrödinger equation with a position dependent or effective mass (EMSE) has found applications in the fields of material science and condensed matter physics, such as semiconductors [1], quantum wells and quantum dots [2,3], 3He clusters [4], graded alloys and semiconductor heterostructures [5-13] etc. It has also been found that such equations appear in very different areas. For example, it has been shown that constant mass Schrödinger equations in curved space and those based on deformed commutation relations can be interpreted in terms of EMSE [14-15]. The position dependent effective mass also appear in non-linear oscillator [16] and $\mathcal{PT}$ symmetric cubic anharmonic oscillator [17]. This has generated a lot of interest in this field and during the past few years various theoretical aspects of EMSE e.g. exact solvability [18-24], shape invariance property [25,26], quasi-exact solvability [27], connection to higher dimensional systems [28], supersymmetric or intertwining formulation [29-32], Lie algebraic approach [33,34], Green's functions [35] etc. have been studied widely. Also the effect of space dependent mass on the revival phenomena [36] and time evolution of wave packets [37] have also been studied. But so far our knowledge goes, the coherent states [38] of an EMSE has not been discussed in the literature. Motivated by the fact that the coherent states for the constant mass Schrödinger equation have revealed a surprisingly rich structure, in this note we shall construct coherent state of an EMSE with harmonic oscillator spectrum by utilising supersymmetric quantum mechanics [39-41] based raising and lowering operators. In this context it should be mentioned that supersymmetric quantum mechanics based raising and lowering operators have found significant application in the construction of coherent states of constant mass Schrödinger equation for various potentials [42-44]. We shall examine different properties of the EMSE coherent states. In particular we shall examine the behaviour of such a state with respect to the physical $x - p$ uncertainty relation. The possibility of squeezing and the effect of variation of mass on it will also be examined. Finally we shall compute the Wigner function and show that it takes negative values in certain ranges of the parameters.

2 Effective mass harmonic oscillator

It may be noted that in the case of an effective mass the kinetic energy (and consequently the Hamiltonian) can be defined in several ways. The most general Hamiltonian can be wriiten in the form [45]

$$H = \frac{1}{4}\left(m^{\alpha}(x)pm^{\beta}(x)pm^{\gamma}(x) + m^{\gamma}(x)pm^{\beta}(x)pm^{\alpha}(x)\right) + V(x) \tag{1}$$

where α, β and γ are parameters satisfying the constraint $\alpha + \beta + \gamma = -1$. Clearly there are different Hamiltonians depending on the choices of the parameters. Here we shall work with the BenDaniel-Duke form which corresponds to the choice $\alpha = \gamma = 0, \beta = -1$ [46]. In this case the Hamiltonian is invariant under instantaneous Galilean transformation [47]. The corresponding Schrödinger equation is given by

$$-\frac{d}{dx}\left(\frac{1}{2m(x)}\frac{d\psi(x)}{dx}\right) + V(x)\psi(x) = E\psi(x) \tag{2}$$

The above equation can be solved in many ways e.g, supersymmetric methods [29-32] , using point canonical transformations [48-50] etc. Here we shall follow the former method and consider the operators

$$A\psi = \frac{1}{\sqrt{2m}}\frac{d\psi}{dx} + W\psi \tag{3}$$

$$A^{\dagger}\psi = -\frac{d}{dx}\left(\frac{\psi}{\sqrt{2m}}\right) + W\psi$$

$$\tag{4}$$

where the function $W(x)$ is known as the superpotential. Then the Hamiltonians

$$H = A^{\dagger}A = -\frac{1}{2m}\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)'\frac{d}{dx} - \left(\frac{W}{\sqrt{2m}}\right)' + W^2 \tag{5}$$

$$\tilde{H} = AA^{\dagger} = -\frac{1}{2m}\frac{d^2}{dx^2} - \left(\frac{1}{2m}\right)'\frac{d}{dx} - \left(\frac{W}{\sqrt{2m}}\right)' + W^2 + \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)'' \tag{6}$$

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are isospectral and the corresponding potentials V and $\tilde{V}$ are given by

$$V = -\left(\frac{W}{\sqrt{2m}}\right)' + W^2 \tag{7}$$

$$\tilde{V} = -\left(\frac{W}{\sqrt{2m}}\right)' + W^2 + \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)'' \tag{8}$$

Then from (5) and (6) it follows that

$$[A, A^\dagger] = \frac{2W'}{\sqrt{2m}} - \left(\frac{1}{\sqrt{2m}}\right)\left(\frac{1}{\sqrt{2m}}\right)'' \tag{9}$$

Clearly if A and $A^\dagger$ are to be interpreted in the same way as the standard harmonic oscillator annihilation and creation operator respectively then we should have $[A, A^\dagger] = 1$ and in this case we obtain from (9)

$$2W(x) = \left(\frac{1}{\sqrt{2m}}\right)' + \int^x \sqrt{2m(z)}dz \tag{10}$$

Thus for a given mass $m(x)$ the superpotential $W(x)$ can be determined from the relation (10). Thus for such superpotentials $A^\dagger$ and A have properties same as the standard bosonic creation and annihilation operators respectively (for example, $A^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}$, $A\psi_n = \sqrt{n}\psi_{n-1}$). In this case the spectrum of $H = A^\dagger A$ and $\tilde{H} = AA^\dagger$ are esentially that of the constant mass harmonic oscillator. In particular the spectrum and eigenfunctions of H are given by

$$E_n = \left(n + \frac{1}{2}\right) \quad , \quad \psi_n(x) = \frac{1}{\sqrt{\sqrt{2\pi}2^n n!}} [2m(x)]^{1/4} \, e^{-\bar{x}^2/4} H_n(\bar{x}/\sqrt{2}) \tag{11}$$

where

$$\bar{x} = \int^x \sqrt{2m(y)} \, dy \tag{12}$$

3   Coherent state and it's properties

There are a number of ways to construct coherent states. However, in view of the fact that the operators A and $A^\dagger$ satisfy the relation $[A, A^\dagger] = 1$, the coherent state can be constructed

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using the displacement operator technique. Thus we define the unitary displacement operator $D(z)$ as

$$D(z) = e^{(zA^\dagger - z^*A)} \tag{13}$$

and the coherent state is given by

$$|\psi\rangle_{cs} = e^{(zA^\dagger - z^*A)}|0\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{n!}|n\rangle \tag{14}$$

Now using (11) and the properties of Hermite polynomials, the coherent state in coordinate representation is found to be [1]

$$\psi_{cs}(x) = \frac{1}{\sqrt{\sqrt{2\pi}}} e^{-(|z|^2/2 - z^2/2)} [2m(x)]^{\frac{1}{4}} e^{-(\bar{x} - 2z)^2/4} \tag{15}$$

In this context it must be mentioned that this formalism of constructing coherent states works only when Eqns.(7) and (10) holds, i.e. the formalism is not suitable for constructing coherent state for a general potential. More specifically, the coherent states can be constructed using the present formalism only when the system has Heisenberg-Weyl, SU(2) or SU(1,1) symmetry. For systems whose symmetry structure is different from these mentioned before, a convenient way to construct coherent state is to follow the Gazeau-Klauder formalism [52,53] which requires the knowledge of the spectrum and the eigenfunctions.

It is not difficult to show that (15) shares many of the properties of standard coherent states. For example, the time dependent coherent state is given by

$$\psi_{cs}(x, t) = e^{-iHt}\psi_{cs}(x) = e^{-(|z|^2 - z'^2 + it)/2}\sqrt{2m(x)}\, e^{-(\bar{x} - 2z')^2/4} \tag{16}$$

where $z' = ze^{-it}$. Thus the coherent state at $t = 0$ remains a coherent state at $t \neq 0$ with a different parameter $z' = ze^{-it}$.

We shall now examine the possibility of squeezing. To this end we consider two Hermitian operators

$$X = \frac{A + A^\dagger}{\sqrt{2}} \quad, \quad Y = \frac{-i(A - A^\dagger)}{\sqrt{2}} \tag{17}$$

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[1]The coherent state (15) can also be obtained as an eigenstate of the annihilation operator A.

where $\Delta X =< X^2 > - < X >^2$. Then using the properties of $A^\dagger, A$ it can be easily shown that

$$(\Delta X) = \frac{1}{2} \ , \ \ (\Delta Y) = \frac{1}{2} \tag{18}$$

Thus the coherent state (15) saturates the uncertainty relation

$$(\Delta X)(\Delta Y) \geq \frac{1}{4} \tag{19}$$

and is a minimum uncertainty state with respect to the relation (19). However, the operators X and Y are not physical position (x) or momentum (p) operators. Thus it would be interesting to examine the behaviour of (15) with respect to the physical $x - p$ uncertainty relation

$$(\Delta x)(\Delta p) \geq \frac{1}{4} \tag{20}$$

We note that if Δx and Δp are each greater than $1/2$, then the above inequality is always true. However, the above inequality holds even if one of Δx or Δp is less than $1/2$ and the other sufficiently large. In this case the state is squeezed. To get a quantitative measure of squeezing we introduce squeezing parameters S_x and S_p defined by :

$$S_x = 2(\Delta x) - 1 \ , \ \ S_p = 2(\Delta p) - 1 \tag{21}$$

Thus the state would be squeezed if either $S_x < 0$ or $S_p < 0$.

We shall now study how far the space dependent mass $m(x)$ influences various features of the coherent state.

Case 1. Let us consider the following mass profile which is considered by some authors in graded alloys [51]

$$2m(x) = cosh^2(\alpha x), \ \ -\infty < x < \infty \tag{22}$$

so that for $\alpha = 0$ we recover the constant mass harmonic oscillator. In this case we have

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{2\pi}2^n n!}} \sqrt{cosh(\alpha x)} \ e^{-\bar{x}^2/4} H_n(\bar{x}/\sqrt{2}) \tag{23}$$

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and

$$\bar{x} = \frac{sinh(\alpha x)}{\alpha}, \quad -\infty < \bar{x} < \infty \tag{24}$$

and the effective mass coherent state is found to be

$$\psi_{cs}(x) = \frac{1}{\sqrt{\sqrt{2\pi}}} e^{-(|z|^2/2 - z^2/2)} \sqrt{cosh(\alpha x)} \, e^{-(sinh(\alpha x)/\alpha - 2z)^2/4} \tag{25}$$

Next we shall examine a very important aspect of the coherent states, namely their behaviour with respect to the physical $x - p$ uncertainty relation. To this end we have evaluated the uncertainty product $(\Delta x)(\Delta p)$ and the squeezing parameters S_x, S_p for the state (25) for different values of the mass parameter α. From Fig. 1a) we find that for larger values of α, the uncertainty product is larger for smaller values of z but for larger values of z, it stabilises and is nearly equal for both values of α. However it always remains greater than 0.25 and consequently the inequality (20) holds.

From Fig. 2(a) it is seen that S_x is negative for some values of the parameters. In particular for fixed α, squeezing increases for larger z. Thus the coherent state exhibits squeezing in the x quadrature. We have plotted S_p in Fig. 3(a). From Fig. 3(a) we find that for larger α, S_p is smaller for smaller z. Subsequently it increases as z increases. However for all values of the parameters $S_p > 0$ implying absence of squeezing in the p quadrature. The same pattern can also be observed for other values of α and z. We note that this non classical behaviour is quite different from the standard harmonic oscillator coherent states (which are minimum $x - p$ uncertainty states and never shows squeezing).

We shall now compute the Wigner function for the coherent state (25). The Wigner function is defined as

$$W(x, p) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_{cs}^*(x - y) e^{2ipy} \psi_{cs}(x + y) \, dy \tag{26}$$

We recall that the Wigner function may or may not take negative values. However, negativity of the Wigner function is a sufficient condition for the state to be nonclassical. In Fig. 4(a)

we have plotted the Wigner function $W(x, p)$ against α and z. It can be seen from the Fig. 4(a) that the Wigner function does take negative values and it confirms non classical nature of the coherent state (25).

Case 2. We now consider another mass profile given by the following which is found to be useful for studying transport properties in semiconductors [29-32,54] given by

$$2m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^2, \quad -\infty < x < \infty \tag{27}$$

In this case

$$\bar{x} = x + (\alpha - 1)\arctan x, \quad -\infty < \bar{x} < \infty \tag{28}$$

The corresponding coherent state is given by

$$\psi_{cs}(x) = \frac{1}{\sqrt{\sqrt{2\pi}}} e^{-(|z|^2 - z^2)/2} \sqrt{\frac{\alpha + x^2}{1 + x^2}} e^{-(x + (\alpha - 1)\arctan x - 2z)^2/4} \tag{29}$$

It may be noted that in this case the mass distribution has different shapes in the ranges $\alpha < 1$ and $\alpha > 1$ (Fig. 5). Now as in the last case we have evaluated the uncertainty product and from Fig. 1(b) we find that the inequality (20) holds. So the coherent state (29) is not a $x - p$ minimum uncertainty state. However, for fixed α, the uncertainty product gets smaller as z increases. So for very large z it behaves like a minimum uncertainty state.

We now compute the squeezing parameters. From Fig. 2(b) and Fig. 3(b) we find that S_x and S_p behave differently for $\alpha > 1$ and $\alpha < 1$. For $\alpha > 1$, S_x starts from a relatively small positive value and S_p starts from a small negative value. As z increases both S_x and S_p stabilize while retaining their character. For $\alpha < 1$ the scenario is exactly opposite. Interestingly in both the ranges of α, S_x remains positive while S_p is always negative. So in this case the coherent state exhibits squeezing in the p quadrature but not in the x quadrature.

We now compute the Wigner function using (26) with $\psi_{cs}(x)$ given by (29). It can be seen from Fig 4(b) that the Wigner function does take a small negative value indicating the non classical nature of the state (29).

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4 Conclusion

Here we have constructed coherent state of effective mass harmonic oscillator with two different mass distributions. It has been shown that in both the cases the coherent states exhibit squeezing and the non classical nature of these states is confirmed by the negativity of the corresponding Wigner functions. In the second case, since the mass function is an increasing function of α, squeezing can be increased using larger α. Let us note that in the case of EMSE coherent state, the displacement operator coherent state and the annihilation operator coherent state are the same as in the case of coherent states of harmonic oscillator in constant mass Schrödinger equation. But, unlike the constant mass case, the EMSE coherent state is not a $x - p$ uncertainty state. It exhibits squeezing. The effect of variation of mass function on the coherent state properties are evident from the figures (1) - (4). Considering the fact that coherent states for systems other than the harmonic oscillator in constant mass Schrödinger equation have attracted much attention for several years [55-71], it would be interesting to construct coherent states for non harmonic type effective mass systems. In such cases it would be useful to carry out the construction using, for example, the Gazeau-Klauder formalism [52,53].

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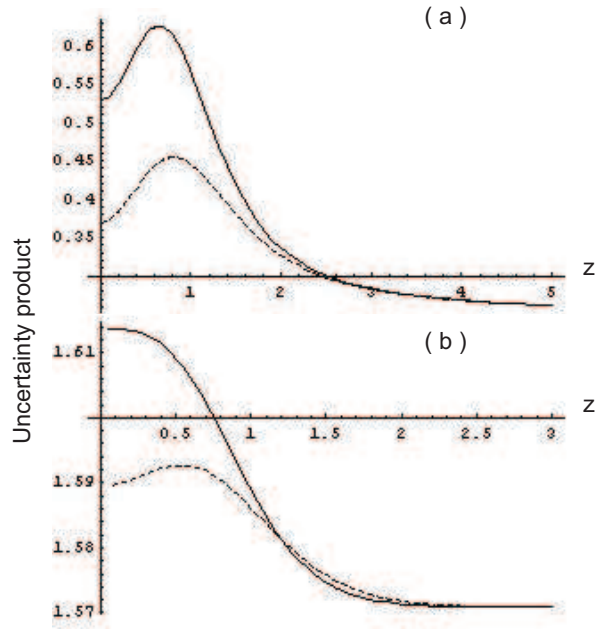
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Fig. 1. (a) Uncertainty product for α = 1.5 (solid curve) and α = 1 (dotted curve) against z.
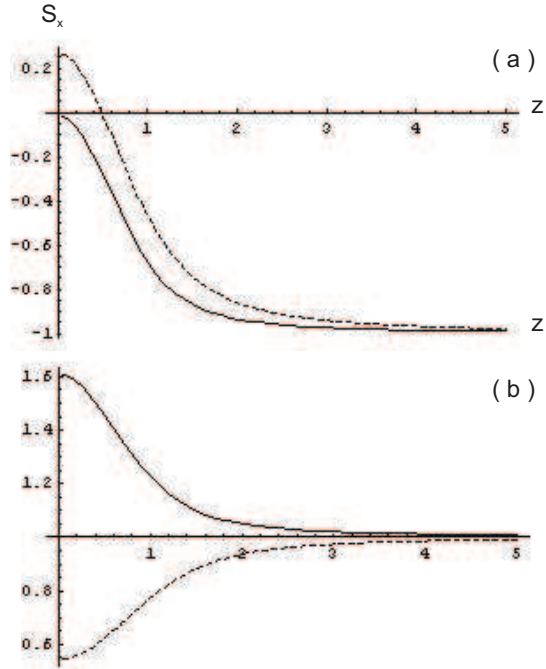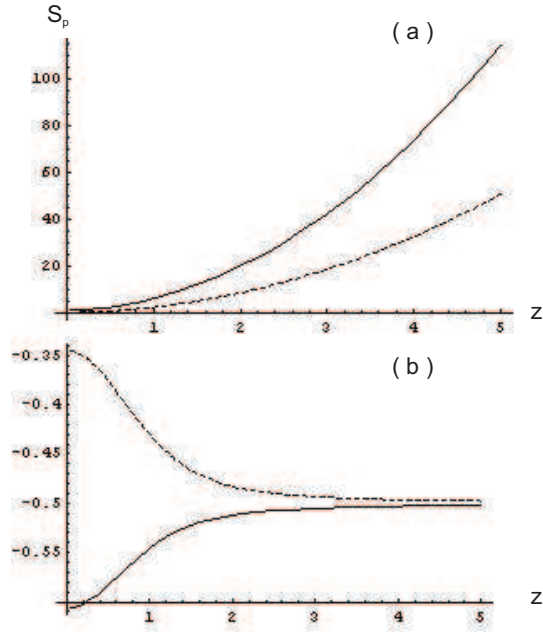(b) Uncertainty product for α = 0.8 (solid curve) and α = 1.2 (dotted curve).

Fig. 2. (a) Graph of S_x against z for α = 1.5 (solid curve) and α = 1 (dotted curve).
(b) Graph of S_x against α =0.8 (solid curve) and α = 1.2 (dotted curve).

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Fig. 3. (a) Graph of S_p against z for α = 1.5 (solid curve) and α = 1 (dotted curve).
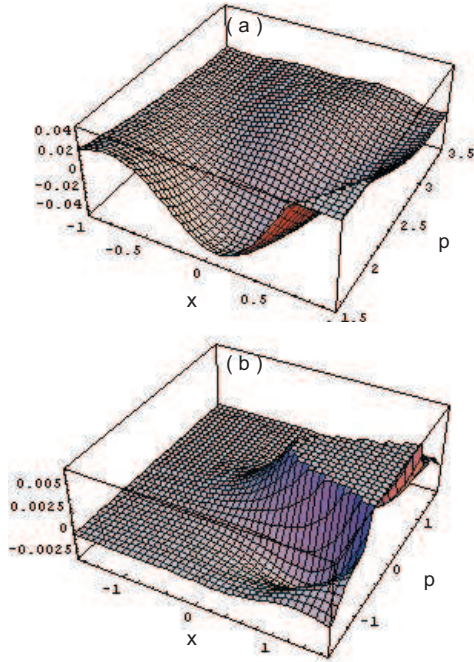(b) Graph of S_p against α =0.8 (solid curve) and α = 1.2 (dotted curve).

Fig. 4. (a) Wigner function of the coherent state (24) for α = 1.2 and z = 0.2.
(b) Wigner function of the coherent state (24) for α = 0.5 and z = 1.5.
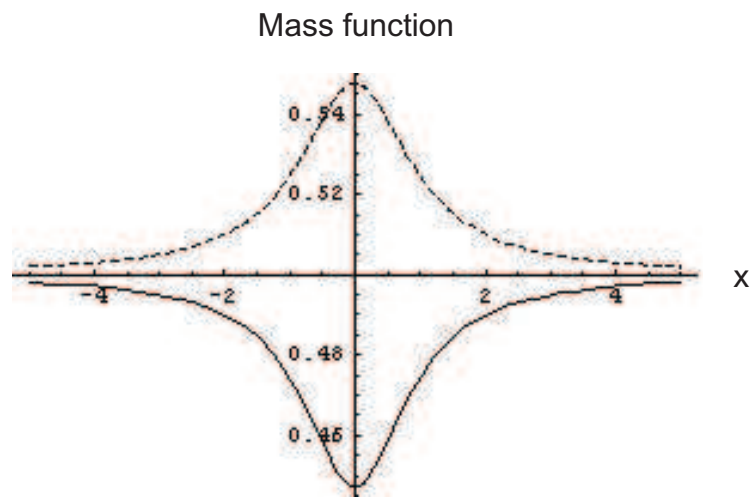
Mass function

Fig. 5. Profile of the mass function (26) for $\alpha = 0.8$ (solid curve) and $\alpha = 1.2$ (dotted curve).